Question 1	Question 2	Question 3	Question 4	Sum	Final score

Written exam ('terzo appello') of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica - 8 July 2013.

SURNAME .... NAME .... MATRICOLA ....

PLEASE NOTE. During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered. Marked tests will be handed out in room 1C150 on 12 July 2013 at 14:30.

Duration: 150 minutes

Question 1.

Let $\alpha, \beta > 0$ and $f : (0, \infty) \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \left(\frac{x}{e}\right)^{\alpha}, & \text{if } \ 0 < x \leq e, \\ \frac{1}{\log^{\beta} x}, & \text{if } \ x > e \end{cases}$$

(i) Find all values of $\alpha, \beta > 0$ such that f has the weak derivative f'_w in $(0, \infty)$ (give a detailed motivation).

(ii) Find all values of $\alpha, \beta > 0$ and $p \in [1, \infty[$ such that $f \in W^{1,p}(0, \infty)$.

(iii) Find all values of $\alpha, \beta > 0$ and $p \in [1, \infty[$ such that $f \in w^{1,p}(0, \infty)$.

Answer:

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Question 2.

(i) Give at least two equivalent definitions of weak derivative.

(ii) State the theorem concerning the characterization of the functions in the Sobolev space $W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^N$, via functions which are 'regular' enough along suitable lines.

(iii) Assume that $p \in [1, \infty[$. Prove that in general $C^\infty(\bar{\Omega})$ is not dense in $W^{l,p}(\Omega)$ (it is enough to give a counterexample and explain it in detail).

Answer:

Question 3.

(i) Give the Definition of cone in $\mathbb{R}^N$ with height h and radius r, and the definition of open set satisfying the cone condition.

(ii) State in detail the theorem which allows to decompose open sets satisfying the cone condition into open sets which are star-shaped with respect to a ball.

(iii) State the Sobolev Integral Representation Theorem.

Answer:

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Question 4.

(i) State the Gagliardo and the Poincaré inequalities.

(ii) Assuming that the Poincaré Inequality holds in the interval (a, b) with $-\infty < a < b < \infty$, prove that the Poincaré Inequality holds in the open subset Ω of $\mathbb{R}^N$ given by $\Omega = \mathbb{R}^{N-1} \times (a, b)$.

(iii) State the Extension Theorem for functions in the Sobolev space $W^{l,p}(\Omega)$.

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